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DIFFRACTION OF A FLEXURAL WAVE BY A SHORT JOINT OF SEMI-INFINITE ELASTIC PLATES†

I. V. ANDRONOV

St Petersburg

Email: iva@aa2628.spb.edu

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The problem of the flexural vibrations of two semi-infinite elastic plates connected along a section of the boundary (the joint) that is short compared with the wavelength of the incident wave, is considered. The problem is reduced to solving integral equations on the section. The use of Green's formula leads to an integral equation with a smooth kernel, the solution of which is a function with singularities of order $-3/2$ at the ends of the section. Regularization of this integral equation is carried out. The asymptotic form of the far field over the dimensionless length of the joint is found. © 2002 Elsevier Science Ltd. All rights reserved.

Approximate models for describing the vibrations of a thin elastic plate are normally derived for a thin infinitely long layer and then extended to plates of finite dimensions. Generally speaking, this leads to an incorrect description of edge effects. Thus, in the vicinity of the ends of fine cracks, inclusions and reinforcements, Kirchhoff's model gives infinite values of the stresses and shearing forces [1, 2]. Nevertheless, it is assumed that an incorrect description of the physical processes close to a crack or reinforcement has no significant effect on the far field of displacements. Moreover, the factor for a singularity of the stress in the neighbourhood of the end of a crack or fine reinforcement, which has been termed the stress intensity factor [1], enables one to judge possible fracture of the plate and crack growth. All the above enables the diffraction by the joint of two semi-infinite plates to be investigated in a first approximation using Kirchhoff's model. The boundary-value problem considered in the present paper is "additional" to the problem of diffraction by a crack that is short compared with the wavelength, investigated in [3].

Practically all explicitly solvable problems of the diffraction and propagation of waves in plates (see, for example, [4, 5]) have already been considered by now. The boundary-value problem investigated here can be formulated in the higher order as explicitly solvable. In fact, to solve the fourth-order differential equation describing the flexural vibrations of a plate within the framework of Kirchhoff's theory, values of the function at a separate point were determined (according to embedding theorems [6]). Thus, in a first approximation the short joint can be regarded as a point joint, which enables the solution of the problem to be constructed in quadratures.

The construction of corrections to such a solution requires an examination of an extended joint. By using the Green's function method, the problem can be reduced to integral equations along a section, in which the unknown quantities are the shearing force and the bending moment on the joint. As is well known [2], in the neighbourhood of the crack edge, the shearing force has a non-integrable singularity, i.e. the solution of the integral equation can be classified as a non-integrable function. A technique for the analytical and numerical analysis of integral equations of this form has been developed in [1]. A slightly different approach‡ is used in the present paper.

The use of the saddle-point method for the integral representation of the solution leads to formulae for the pattern of the flexural wave diverging from the joint, while the residues at the poles of the integrand give the edge waves [7] which propagate along the edges of the plates. When constructing the asymptotic form of the radiation pattern and the amplitudes of the edge waves, it is necessary to carry out cumbersome transformations. In view of this, independent monitoring of the final asymptotic formulae is necessary, for which the reciprocity principle and the optical theorem [8] will be used below. Values of the pattern calculated by means of the asymptotic formula are given. It is pointed out that the corrections are considerable for a joint length of the order of 10^{-5} wavelength.

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‡ANDRONOV, I. V., Scattering of flexural waves by thin inhomogeneities in an elastic plate. Deposited at the All-Union Institute of Scientific and Technical Information, No. 66-V90, Moscow, 1990.

1. FORMULATION OF THE DIFFRACTION PROBLEM

We will consider a system of two semi-infinite plates $\Pi_\pm = \{-\infty < x < +\infty, \pm y > 0\}$ with free edges $\{y = \pm 0\}$, connected along the section $\Lambda = \{|x| \leqslant a, y = 0\}$. Vibrations are excited by a plane flexural wave

$$\xi^i = \exp(ik_0(x\cos\vartheta_0 - y\sin\vartheta_0))$$

in one of the plates. The total field in the system consists of the incident wave ξ^i, the wave ξ^r reflected from the free edge of the plate and the diffraction field ξ^s. The total field is the solution of the following boundary-value problem

$$\begin{aligned}
&(\Delta^2 - k_0^4)\xi = 0, \quad y \neq 0\\
&\mathbb{F}\xi = 0, \quad \mathbb{M}\xi = 0, \quad |x| > a, y = 0\\
&[\xi] = 0, \quad [\xi_y] = 0, \quad [\mathbb{F}\xi] = 0, \quad [\mathbb{M}\xi] = 0, \quad |x| < \alpha\\
&\mathbb{F}\xi \equiv \xi_{yyy} + (2-\sigma)\xi_{xxy}, \quad \mathbb{M}\xi \equiv \xi_{yy} + \sigma\xi_{xx}
\end{aligned} \tag{1.1}$$

where k_0 is the wave number of the flexural waves, operators of the shearing force $\mathbb{F}$ and the bending moment $\mathbb{M}$ are introduced on the line y = const, σ is Poisson's ratio and $[f]$ is the abrupt change in the function f on the line $y = 0$.

The scattered field satisfies the radiation principle, which can be defined specifically in the form of the prescribed asymptotic form of the far field. The scattered field ξ^s forms, at considerable distances from the joint Λ, a diverging wave with asymptotic form

$$\xi^s \sim \sqrt{\frac{2\pi}{k_0 r}} \exp\left(ik_0 r - i\frac{\pi}{4}\right)\Psi(\vartheta), \quad r \to +\infty, \quad 0 < \varepsilon \leqslant |\vartheta| \leqslant \pi - \varepsilon \tag{1.2}$$

and four edge waves [7, 8] having the asymptotic forms

$$\xi^s_{x\to\pm\infty} \sim (A^e_\pm + \operatorname{sign}(y)A^0_\pm)\exp(\pm i\varkappa x)\Big\{\exp(-\sqrt{\varkappa^2 - k_0^2}\,|y|) - \\ -\frac{(1-\sigma)\varkappa^2 - k_0^2}{(1-\sigma)\varkappa^2 + k_0^2}\exp(-\sqrt{\varkappa^2 + k_0^2}\,|y|)\Big\}, \quad |y| \leqslant y^* \tag{1.3}$$

Here the edge waves are represented in the form of the sum of symmetrical waves with amplitudes $A^e_\pm$ and antisymmetrical waves with amplitudes $A^o_\pm$. The wave number $\varkappa$ of the edge waves is defined by the formula [7, 8]

$$\varkappa = k_0((1-\sigma)(3\sigma - 1 + 2\sqrt{1 - 2\sigma + 2\sigma^2}))^{-1/4}$$

The $\Psi(\vartheta)$ diagram and the amplitudes of the edge waves $A_\pm$ in the asymptotic forms are assumed to be finite. Note that the $\Psi(\vartheta)$ diagram is a meromorphic function in the complex plane ϑ.

Below, the quantity $k_0 a$ will be assumed to be small.

2. THE GEOMETRIC PART OF THE FIELD

We will consider the problem for unjoined plates. Then conditions (1.1) are satisfied along the entire axis and the plates vibrate independently. The filed ξ^r consists of a reflected plane wave and a non-uniform wave

$$\xi^r = R(\vartheta_0)\exp(ik_0(x\cos\vartheta_0 + y\sin\vartheta_0)) + T(\vartheta_0)\exp(ik_0 x\cos\vartheta_0 - y\sqrt{1+\cos^2\vartheta_0})$$

The reflection coefficient $R(\vartheta_0)$ and the transformation coefficient $T(\vartheta_0)$ are determined from boundary conditions (1.1). Solving the system of two equations, we obtain

$$R(\vartheta_0)=-\frac{\overline{L(\vartheta_0)}}{L(\vartheta_0)},\quad T(\vartheta_0)=-\frac{2i\sin\vartheta_0 A_+(\vartheta_0)A_-(\vartheta_0)}{L(\vartheta_0)}$$

$$L(\vartheta_0)=i\sin\vartheta_0 A_+^2(\vartheta_0)+\sqrt{1+\cos^2\vartheta_0}\,A_-^2(\vartheta_0),\quad A_\pm(\vartheta_0)=(1-\sigma)\cos^2\vartheta_0\pm 1$$

We will define the geometric part of the field as follows:

$$\xi^g=\begin{cases}\xi^i+\xi^r, & y>0\\ 0, & y<0\end{cases}$$

and we will calculate the abrupt changes in ξ^g along the section λ. It is obvious that only the displacements and angles of inclination have discontinuities, while the moments and forces are continuous owing to conditions (1.1) being satisfied for $\xi^i+\xi^r$, including on the section Λ. We have

$$[\xi^g]=-\frac{4i\sin\vartheta_0}{L(\vartheta_0)}A_+(\vartheta_0)\exp(ik_0x\cos\vartheta_0)$$

$$[\xi_y^g]=-\frac{4ik_0\sqrt{1+\cos^2\vartheta_0}\,\sin\vartheta_0}{L(\vartheta_0)}A_-(\vartheta_0)\exp(ik_0x\cos\vartheta_0)$$

3. CHANGE TO INTEGRAL EQUATIONS

Abrupt changes in the geometric part of the field are compensated by the correction ξ^s – the scattered field. Consider Green's formula [9] for the scattered field ξ^s and Green's function $G(x,y,x_0,y_0)$ for an infinite plate Π_+ with a free edge. Such a Green's function can be constructed by the method of separation of variables and was investigated earlier [10]. The asymptotic forms (1.2) and (1.3) lead to the disappearance of integrals over an arc of large radius, and for the field ξ^s in plate Π_+ the following integral representation is obtained

$$\xi^s(x_0,y_0)=\int_{-a}^{a}(G(x,0,x_0,y_0)\mathbb{F}\xi^s(x,0)-G_y(x,0,x_0,y_0)\mathbb{M}\xi^s(x,0))dx,\quad y_0>0$$

The representation for the field in plate Π_- is obtained in a similar way. We will write both formulae together

$$\xi^s(x_0,y_0)=\operatorname{sign}(y_0)\int_{-a}^{a}G(x,0,x_0,|y_0|)p(x)dx+\int_{-a}^{a}G_y(x,0,x_0,|y_0|)q(x)dx \tag{3.1}$$

The functions $p(x)$ and $q(x)$ denote (apart from a factor) the force and moment of the total field at the joint Λ.

Representation (3.1) was obtained formally. Close to the ends of thin inhomogeneities the field has singularities and the shearing force may be non-integrable [1]. An investigation was made in [2] of the field of displacements in the neighbourhood of the end of a semi-infinite crack and it was shown that the shearing force has a singularity of the order of $-3/2$. Thus, function $p(x)$ in integral representation (3.1) has singularities of the form $(x\pm a)^{-3/2}$, and the kernel of the integral equation for $p(x)$ is a continuous function. Methods have been developed [1] for regularizing integral representations and integral equations of this kind, and also schemes for their numerical solution.

We will consider in more detail the causes of the emergence of a non-integrable weight $p(x)$ in integral representation (3.1). We will seek the part of the field that is uneven with respect to y by the Fourier transformation method. As a consequence of the radiation conditions, we have

$$\xi^o(x_0,y_0)=\operatorname{sign}(y_0)\int e^{i\lambda x_0}(\alpha^-(\lambda)e^{-b_-(\lambda)|y_0|}+\alpha^+(\lambda)e^{-b_+(\lambda)|y_0|})\frac{d\lambda}{l(\lambda)} \tag{3.2}$$

$$l(\lambda)=a_+^2(\lambda)b_-(\lambda)-a_-^2(\lambda)b_+(\lambda),\quad a_\pm(\lambda)=(1-\sigma)\lambda^2\pm k_0^2,\quad b_\pm=\sqrt{\lambda^2\pm k_0^2}$$

The factor $1/l(\lambda)$ is introduced for convenience.

Boundary conditions (1.1) impose certain constraints on the functions $\alpha^{\pm}(\lambda)$. In view of the unevenness of the field ξ^0 and its continuity on Λ, the condition that the moment $\mathbb{M}\xi^0$ equals zero is satisfied on the entire axis and can easily be inverted. As a result, the functions $\alpha^{\pm}(\lambda)$ can be expressed in terms of a single unknown function $\alpha(\lambda)$

$$\alpha^-(\lambda) = a_+(\lambda)\alpha(\lambda), \quad \alpha^+(\lambda) = -a_-(\lambda)\alpha(\lambda)$$

From the boundary condition for the force we obtain the equation

$$\int e^{i\lambda x}\alpha(\lambda)d\lambda = 0, \quad |x| > a \tag{3.3}$$

The continuity of the field ξ on Λ gives the paired equation

$$2k_0^2 \int e^{i\lambda x}\frac{\alpha(\lambda)}{l(\lambda)}d\lambda = -\frac{1}{2}[\xi^g](x), \quad |x| < a \tag{3.4}$$

Paired equations (3.3) and (3.4) uniquely define the function $\alpha(\lambda)$ in the class of functions with no more than linear growth at infinity, i.e. $\alpha(\lambda) = O(\lambda)$.

If Eq. (3.3) is formally inverted, the following representation is obtained for the function $\alpha(\lambda)$

$$\alpha(\lambda) = \int_{-a}^{a} e^{-i\lambda x}p(x)dx$$

The function $p(x)$ is the same as in integral representation (3.1). Since the function $\alpha(\lambda)$ increases at infinity, its Fourier transform $p(x)$ contains non-integrable singularities. Changing the order of integration in representation (3.2) and calculating the integral over λ, we formally obtain representation (3.1).

To eliminate non-integrable singularities of $p(x)$, we will represent the function $\alpha(\lambda)$ in the form

$$\alpha(\lambda) = \lambda^2\hat{\alpha}(\lambda) + \alpha_0\rho_0(\lambda) + \alpha_1\rho_1(\lambda) \tag{3.5}$$

The function $\hat{\alpha}(\lambda)$ decreases at infinity no more slowly than $O(\lambda^{-1})$, but the remaining two terms compensate for the double zero at $\lambda = 0$ of the first term. As the functions $\rho_0(\lambda)$ and $\rho_1(\lambda)$ one can choose any functions that do not contradict the asymptotic form of $\alpha(\lambda)$ at infinity, and are such that the vectors $(\rho_0(0), \rho_0'(0))$ and $(\rho_1(0), \rho_1'(0))$ are linearly independent. Furthermore, the carriers of the Fourier transformation of the functions $\rho_0(\lambda)$ and $\rho_1(\lambda)$ must belong to the interval $|x| < a$.

In taking into account representation (3.5) and the above properties of the functions $\rho_0(\lambda)$ and $\rho_1(\lambda)$, Eq. (3.3) is inverted and yields

$$\alpha(\lambda) = \lambda^2 \int_{-a}^{a} e^{-i\lambda x}\hat{p}(x)dx + \alpha_0\rho_0(\lambda) + \alpha_1\rho_1(\lambda) \tag{3.6}$$

Now, owing to the asymptotic form $\hat{\alpha}(\lambda) = O(\lambda^{-1})$, the function $\hat{p}(x)$ at the ends of the integration interval vanishes, i.e. the following representation of this function holds

$$\hat{p}(x) = \sqrt{a^2 - x^2}\,P(x) \tag{3.7}$$

where $P(x)$ is a bounded function.

Substituting expression (3.6) into Eq. (3.4), and changing the order of integration, we obtain an integral-algebraic equation for determining the function $\hat{p}(x)$ and the constants α_0 and α_1

$$\int_{-a}^{a} \hat{p}(x)G_{xx}(x,0,x_0,0)dx - k_0^2 \int e^{i\lambda x_0}\frac{\alpha_0\rho_0(\lambda) + \alpha_1\rho_1(\lambda)}{l(\lambda)}\lambda^2 d\lambda = \frac{1}{4}[\xi^g](x_0) \tag{3.8}$$

where

$$G_{xx}(x,0,x_0,0) = k_0^2 \int e^{i\lambda(x-x_0)}\frac{(i\lambda)^2}{l(\lambda)}d\lambda$$

For the even part of the scattered field ξ^e it is possible to retain representation (3.1), since the unknown quantity $q(x)$ is integrable. Taking into account the expression for Green's function, we conclude that this representation takes the form

$$\xi^e(x_0, y_0) = \int (a_-(\lambda)b_+(\lambda)e^{-b_-(\lambda)|y_0|} - a_+(\lambda)b_-(\lambda)e^{-b_+(\lambda)|y_0|}) \int_{-a}^{a} e^{i\lambda(x-x_0)} q(x)dx \frac{d\lambda}{l(\lambda)} \tag{3.9}$$

Substituting expression (3.9) into the boundary condition $[\xi_y] = 0$, we obtain an integral equation in $q(x)$

$$\int_{-a}^{a} q(x) G_{yy_0}(x, 0, x_0, 0)dx = \frac{1}{4}[\xi_y^g](x_0) \tag{3.10}$$

$$G_{yy_0}(x, 0, x_0, 0) = k_0^2 \int e^{i\lambda(x-x_0)} \frac{\sqrt{\lambda^4 - k_0^4}}{l(\lambda)} d\lambda$$

4. INVESTIGATION OF THE KERNELS OF THE INTEGRAL EQUATIONS

The solvability of the equations is determined by the singularities of the kernels. To determine the asymptotic form of the function $G(x, 0, x_0, 0)$, $G_{xx}(x, 0, x_0, 0)$ and $G_{yy_0}(x, 0, x_0, 0)$ as $|x - x_0| \to 0$, we will investigate the behaviour of the integrands for large λ. We can verify the correctness of the asymptotic form

$$l(\lambda) = k_0^2 \chi |\lambda|^3 + \ldots, \quad \chi = (1-\sigma)(3+\sigma)$$

Hence Green's function $G(x, 0, x_0, 0)$ is finite when $x = x_0$, and its second derivatives $G_{xx}(x, 0, x_0, 0)$ and $G_{yy_0}(x, 0, x_0, 0)$ have logarithmic singularities.

To prove the theorems of the existence and uniqueness of the solutions of integral equations with a logarithmic kernel, it is necessary to calculate the index of the integral operator [11]. Investigating the Fourier transforms of the kernels G_{xx} and G_{yy0}, we establish their sectoriality: if irrationalities in the denominator are eliminated, it is possible to note that the imaginary part of the Fourier transforms does not change sign, and consequently the kernels possess the property of sectoriality and the index is equal to zero.

Using the theorems of the existence and uniqueness of solutions [12], we conclude the integral equation (3.10) has a unique solution, which can be represented in the form

$$q(x) = \frac{Q(x)}{\sqrt{a^2 - x^2}}, \quad Q(x) \in C \tag{4.1}$$

Integral equation (3.8) with any prescribed α_0 and α_1 can be solved uniquely in class (4.1). However, the solution $\hat{p}(x)$ may vanish at the ends of the integration interval, which is achieved by the choice of the constants α_0 and α_1.

We will continue the investigation of the kernels of the integral equations. To obtain the asymptotic form of the radiation pattern of the scattered field in the two higher orders with respect to the asymptotically small parameter k_0a, it is necessary to know the asymptotic form of the kernels up to terms that are quadratic in $x - x_0$.

Owing to the evenness of the function $G(x, 0, x_0, 0)$ with respect to $x - x_0$, we have

$$G(x, 0, x_0, 0) = k_0^{-2} I + \hat{G}(x - x_0)$$

$$\hat{G}''(x - x_0) = G_{xx}(x, 0, x_0, 0) = -\int \frac{e^{ik_0\tau|x-x_0|}\tau^2}{\mathcal{L}(\tau)} d\tau, \quad \hat{G}(0) = \hat{G}'(0) = 0 \tag{4.2}$$

Here

$$I = \int \frac{d\tau}{\mathcal{L}}(\tau), \quad \mathcal{L}(\tau) = k_0^{-5} l(k_0\tau) = A_+^2(\tau)\sqrt{\tau^2-1} - \mathcal{A}_-^2(\tau)\sqrt{\tau^2+1}$$

$$\mathcal{A}_\pm(\tau) = k_0^{-2} a_\pm(k_0\tau) = (1-\sigma)\tau^2 \pm 1$$

To calculate the asymptotic form of the integral in expression (4.2) when $s \to 0$, we will use the integral representation of the MacDonald function

$$2K_0(s) = \int \frac{e^{i\tau s} d\tau}{\sqrt{\tau^2+1}}, \quad 2K_0(is) = \int \frac{e^{i\tau s} d\tau}{\sqrt{\tau^2-1}}$$

and introduce the notation

$$J_{km} = \int \frac{H_m(\tau)\tau^k}{\mathcal{L}(\tau)} d\tau, \quad k = 0, 2; \quad m = 0, 1$$

$$H_m(\tau) = (1-\sigma^{2-m})\tau^2\left(\frac{\tau^2}{\sqrt{\tau^4-1}} - 1\right) - \frac{1-m\sigma(1+\sigma/2)}{\sqrt{\lambda^4-1}}, \quad m = 0, 1$$

We find

$$G_{xx}(x, 0, x_0, 0) = -\frac{1}{\chi}(K_0(k_0|x-x_0|) + K_0(ik_0|x-x_0|)) + \frac{1}{\chi}\int e^{ik_0\tau|x-x_0|} H_0(\tau)\frac{d\tau}{\mathcal{L}(\tau)} \tag{4.3}$$

The values of the integral in formula (4.3) and its derivatives up to the third order when $x = x_0$ can be calculated by direct substitution. Taking into account the asymptotic form of the MacDonald function and integrating, we obtain the asymptotic form (γ is Euler's constant)

$$G(x, 0, x_0, 0) = \frac{I}{k_0^2} + \frac{1}{\chi}(x-x_0)^2 \ln|x-x_0| + \frac{1}{\chi}\left(\ln\frac{k_0}{2} + \gamma + i\frac{\pi}{4} + \frac{J_{00}}{2} - \frac{3}{2}\right)(x-x_0)^2 -$$
$$-\frac{k_0^2}{24\chi}\left(J_{20} + i\frac{\pi}{4}\right)(x-x_0)^4 + O((x-x_0)^6 \ln|x-x_0|)$$

In a similar way we can obtain the asymptotic form of the kernel of integral equation (3.10)

$$G_{yy_0}(x, 0, x_0, 0) = -\frac{2}{\chi}\ln|x-x_0| - \frac{2}{\chi}\left(\ln\frac{k_0}{2} + \gamma + i\frac{\pi}{4} - J_{01}\right) +$$
$$+\frac{k_0^2}{\chi}\left(J_{21} + i\frac{\pi}{8}\right)(x-x_0)^2 + O((x-x_0)^4 \ln|x-x_0|) \tag{4.4}$$

5. THE ASYMPTOTIC FORM OF THE SOLUTIONS OF THE INTEGRAL EQUATIONS

An analysis of the singularities of the kernels of integral-algebraic equation (3.8) and integral equation (3.10) enables us to conclude that the solutions $\hat{p}(x)$, α_0, α_1 and $q(x)$ exist and are unique in the corresponding classes.

To calculate the asymptotic form of the solution of Eq. (3.8) it is convenient to select the functions $\rho_0(\lambda)$ and $\rho_1(\lambda)$ in the form

$$\rho_0(\lambda) = \int_{-a}^{a} \frac{e^{-i\lambda s} ds}{\sqrt{a^2-s^2}}, \quad \rho_1(\lambda) = \int_{-a}^{a} \frac{e^{-i\lambda s} s ds}{\sqrt{a^2-s^2}} \tag{5.1}$$

We will represent the function $\hat{p}(x)$ in the form (3.7) and expand the function $P(x)$ in a Taylor series. To obtain the far-field asymptotic form up to terms $O((k_0a)^2)$, it is sufficient to retain only the two leading terms

$$\hat{p}(x) \approx (P(0) + P'(0)x)\sqrt{a^2 - x^2} \tag{5.2}$$

We substitute expressions (5.1) and (5.2) into Eq. (3.8) and replace the kernels with the asymptotic forms. After replacing the variables $x = as$ and $x_0 = as_0$ and evaluating the integrals, we obtain an algebraic equation in $P(0)$, $P'(0)$, α_0 and α_1. The left-hand side of this equation is a third-degree polynomial with coefficients of even powers which depend on $P(0)$ and α_0, and with coefficients of uneven powers which depend on $P'(0)$ and α_1. The right-hand side can likewise be expanded in a power series s_0. After equating the coefficients of powers of s_0, we obtain a system of linear equations. This system splits into two systems of two equations each: the first system enables us to determine the quantities $P(0)$ and α_0, while the unknown quantities $P'(0)$ and α_1 are determined from the second system. We find

$$\alpha_0 a^2 = -\frac{(k_0a)^2}{4\pi l}[\xi^g](0) - \frac{(k_0a)^4}{16\pi l}[\xi^g](0)(B-1)\cos^2\vartheta_0 -$$
$$-\chi\frac{(k_0a)^4}{16\pi l^2} = [\xi^g](0)(B^2 - 2B + 2)$$

$$P(0)a^2 = -\frac{(k_0a)^2}{8\pi l}[\xi^g](0)B - \frac{(k_0a)^2}{8\pi}\chi[\xi^g](0)\cos^2\vartheta_0.$$

$$\alpha_1 a^3 = \frac{i}{2\pi}\frac{\chi}{B}k_0a[\xi^q]\cos\vartheta_0\left\{1 + \frac{7}{32}(k_0a)^2\frac{J_{20} + i\pi/4}{B}\right\} -$$
$$-\frac{i}{8\pi}\frac{\chi}{B}(k_0a)^3[\xi^q]\cos^3\vartheta_0$$

$$P'(0)a^3 = -\frac{i}{4\pi}\frac{\chi}{B}k_0a[\xi^g]\cos\vartheta_0$$

Here

$$B = 2\ln\frac{k_0a}{4} + 2\gamma + J_{00} + i\frac{\pi}{2}$$

We now consider integral equation (3.10). We represent the solution in the form (4.1) and expand the function $Q(as)$ in a Taylor series. We will confine ourselves to three terms

$$Q(as) = Q_0 + Q_1as + Q_2a^2s^2 + \ldots \tag{5.3}$$

We substitute expressions (5.3) and (4.4) into Eq. (3.10) and, evaluating the integrals, we obtain, as earlier, a system of equations in the coefficients $Q_j (j = 0, 1, 2)$. The equation for Q_1 is removed.

We obtain the asymptotic forms of the coefficients

$$Q_0 = -\frac{\chi}{4\pi A}[\xi_y^g](0) + (k_0a)^2\frac{\chi}{4\pi}[\xi_y^g](0)\left(\left(\frac{1}{2} + \frac{1}{4A}\right)\cos^2\vartheta_0 - \frac{C}{A}\right)$$

$$Q_1 a = ik_0a\frac{\chi}{4\pi}[\xi_y^g](0)\cos\vartheta_0$$

$$Q_2 a^2 = (k_0a)^2\frac{\chi}{2\pi}[\xi_y^g](0)\left(\frac{C}{A} - \frac{1}{2}\cos^2\vartheta_0\right)$$

Here

$$A = \ln\left(\frac{k_0a}{4}\right) + \gamma - 2J_{01} + i\frac{\pi}{2}, \quad C = J_{21} + i\frac{\pi}{8}$$

6. THE FAR FIELD

Evaluating the integrals over λ in (3.2) and (3.9) by the saddle point method [13], we obtain expressions for the $\Psi(\vartheta)$ pattern of the diverging wave (1.2). We have

$$\Psi(\vartheta)=\frac{k_0\sin\vartheta}{l(k_0\cos\vartheta)}\{a_+(k_0\cos\vartheta)\alpha(k_0\cos\vartheta)\operatorname{sign}(y)+$$

$$+\ k_0a_-(k_0\cos\vartheta)\sqrt{1+\cos^2\vartheta}\int\limits_{-a}^{a}q(x)e^{-ik_0x\cos\vartheta}dx\Big\}\tag{6.1}$$

On deforming the integration contour, residues at the zeros of the denominator $l(\lambda)$ are singled out, which gives the edge waves

$$A_\pm^\circ=2\pi i\frac{a_+(\varkappa)\alpha(\pm\varkappa)}{l'(\pm\varkappa)},\quad A_\pm^e=2\pi i\frac{a_-(\varkappa)\sqrt{\varkappa^2+k_0^2}}{l'(\pm\varkappa)}\int\limits_{-a}^{a}e^{\pm i\varkappa x}q(x)dx\tag{6.2}$$

Formulae (6.1) and (6.2) give accurate expressions for the pattern and the amplitudes of the edge waves if the functions $\alpha(\lambda)$ and $q(x)$ occurring in them are calculated accurately.

Note that expressions (6.1) and (6.2) are connected by the relations

$$A_\pm=-2\pi i\operatorname*{Res}_{\vartheta=\arccos(\pm\varkappa/k_0)}\Psi(\vartheta)\tag{6.3}$$

which is similar to that obtained earlier [14] for the problem of the diffraction of an acoustic wave by a thin plate.

Using the asymptotic forms of the solutions of integral equations (3.8) and (3.10) found in the previous section, we obtain the leading terms of the expansion of $\Psi(\vartheta)$ with respect to the small parameter k_0a

$$\Psi(\vartheta,\vartheta_0)=i\frac{\sin\vartheta}{L(\vartheta)}\frac{\sin\vartheta_0}{L(\vartheta_0)}\Big\{\operatorname{sign}(y)A_+(\vartheta)A_+(\vartheta_0)\Big[\frac{1}{I}-\frac{\chi}{B}\cos\vartheta\cos\vartheta_0+$$

$$+\ (k_0a)^2\Big(\frac{B^2-2B+2}{4I^2\chi}+\frac{B-1}{4I}(\cos^2\vartheta+\cos^2\vartheta_0)+\frac{\chi}{4}\cos^2\vartheta\cos^2\vartheta_0+$$

$$+\ \frac{\chi}{4B}\Big(-\frac{7}{8}\frac{J_{20}+i\pi/4}{B}+\cos^2\vartheta+\cos^2\vartheta_0\Big)\cos\vartheta\cos\vartheta_0\Big)+\ldots\Big]+$$

$$+\ \chi A_-(\vartheta)A_-(\vartheta_0)\sqrt{1+\cos^2\vartheta}\sqrt{1+\cos^2\vartheta_0}\Big[\frac{1}{A}+$$

$$+\ (k_0a)^2\Big(\frac{1}{2}\cos\vartheta\cos\vartheta_0+\frac{1}{2A}(\cos^2\vartheta+\cos^2\vartheta_0)\Big)+\ldots\Big]\Big\}\tag{6.4}$$

From formulae (6.2) it is possible to write the asymptotic forms of the edge waves. However, these asymptotic forms can also be obtained from (6.4) using relation (6.3).

7. EVALUATION OF THE INTEGRALS

The asymptotic form of the far-field pattern (6.4) contains the integrals I, J_{00} and J_{01}, and the denominators of their integrands contain the factor $\mathcal{L}(\tau)$. These integrals can be reduced to the sums of residues at the poles, determined by the roots of the function $\mathcal{L}(\tau)$ that lie on the Riemann surface of the root $\sqrt{\tau^4-1}$. We fix the branches of the square roots, making sections in the complex plane τ along the rays

$$L_1=[1,1+i\infty),\quad L_2=[i,-\infty+i),\quad L_3=[-1,-1-i\infty),\quad L_4=[-i,+\infty-i)$$

We will examine the roots of the denominator $\mathcal{L}(\tau)$. They are positioned in pairs on a four-sheet Riemann surface. Four roots lie on a physical sheet:

$$\tau=\tau_0=\varkappa/k_0,\quad \tau=-\tau_0,\quad \tau=\tau_1,\quad \tau=-\tau_1$$

where

$$\tau_1 = \left((1-\sigma)(1-3\sigma+2\sqrt{1-2\sigma+2\sigma^2})\right)^{-1/4} e^{i\pi/4}$$

On the remaining sheets of the Riemann surface, the roots lie at the points $\tau = \pm i\tau_0$ and $\tau = \pm i\tau_1$.

We will first consider the integral I, which occurs in the leading term of the asymptotic form. We deform the integration contour in the upper half-plane and stretch it on sections L_1 and L_2. In this case, the residues are isolated at the poles τ_0 and τ_1. In the integrals over loops C_1 and C_2 covering the rays L_1 and L_2, we change to integration over the right-hand edges. In the integral over L_2 we make the replacement of variable $\tau = it$, which transforms ray L_2 to L_1, and we combine the integrals. It can be established that the integrals cancel out over the sections. Thus,

$$I = 2\pi i\left(\frac{1}{d\mathscr{L}(\tau_0)/d\tau} + \frac{1}{d\mathscr{L}(\tau_1)/d\tau}\right)$$

Note that integral I is a purely imaginary quantity.

We now perform similar transformations with the remaining integrals. In this case, the contributions of the integrals over the sections for J_{20} cancel out as for I, while for J_{00} and J_{01} they are doubled. For J_{20} we have

$$J_{20} = 2\pi i\left(\frac{H_0(\tau_0)\tau_0^2}{d\mathscr{L}(\tau_0)/d\tau} + \frac{H_0(\tau_1)\tau_1^2}{d\mathscr{L}(\tau_1)/d\tau}\right)$$

In the integrals J_{00} and J_{01} the integrals over the ray L_1 are reduced to the sums of the residues using a procedure developed earlier [5]. For this we introduce the function

$$f(\tau) = \ln(\tau - \sqrt{\tau^2-1})$$

It it possible to verify that $f(-\tau) = i\pi - f(\pi)$, which enables the following transformations of the integrals to be carried out

$$\int_1^{1+i\infty} F(\tau^2)d\tau = \frac{1}{2\pi i}\int_{C_1} F(\tau^2)(f(\tau)-f(-\tau))d\tau = \frac{1}{2\pi i}\int_{C_1\cup C_3} F(\tau^2)f(\tau)d\tau$$

Now the integrals from combining the loops C_1 and C_2 can be evaluated as the sums of residues. Finally we have

$$J_{0m} = 2\pi i\left(\frac{H_0(\tau_0)}{d\mathscr{L}(\tau_0)/d\tau} + \frac{H_0(\tau_1)}{d\mathscr{L}(\tau_1)/d\tau}\right) +$$
$$+ \frac{1}{4}\sum_{j=0}^{7} \frac{((1-\sigma^{2-m})\tau_j^4 - 1 + m\sigma(1+\sigma/2))\mathscr{A}_-^2(\tau_j) - (1-\sigma^{2-m})\tau_j^2(\tau_j^2-1)\mathscr{A}_+^2(\tau_j)}{(1-\sigma)((1-\sigma)^2(3+\sigma)\tau_j^4 - 1 + 3\sigma)\tau_j^3\sqrt{\tau_j^2-1}} f(\tau_j)$$

8. THE OPTICAL THEOREM AND NUMERICAL RESULTS

The asymptotic formulae obtained satisfy the reciprocity principle, i.e. the expression for the pattern does not change when ϑ is replaced by ϑ_0 and ϑ_0 is replaced with ϑ. Furthermore, the formulae obtained can be monitored using the optical theorem. Earlier [8], a different normalization of the edge waves was adopted, and an error was allowed in calculating the energy flux transferred by the edge wave.

We will formulate the optical theorem as it applies to the geometry of the system examined here. The effective scattering cross-section, calculated as the proportion of energy taken from the geometric part of the field, is expressed by the formula

$$\Sigma = -\frac{4\pi}{k_0}\operatorname{Re}\left(\overline{R(\vartheta_0)}\Psi(\vartheta_0,\vartheta_0)\right) \tag{8.1}$$

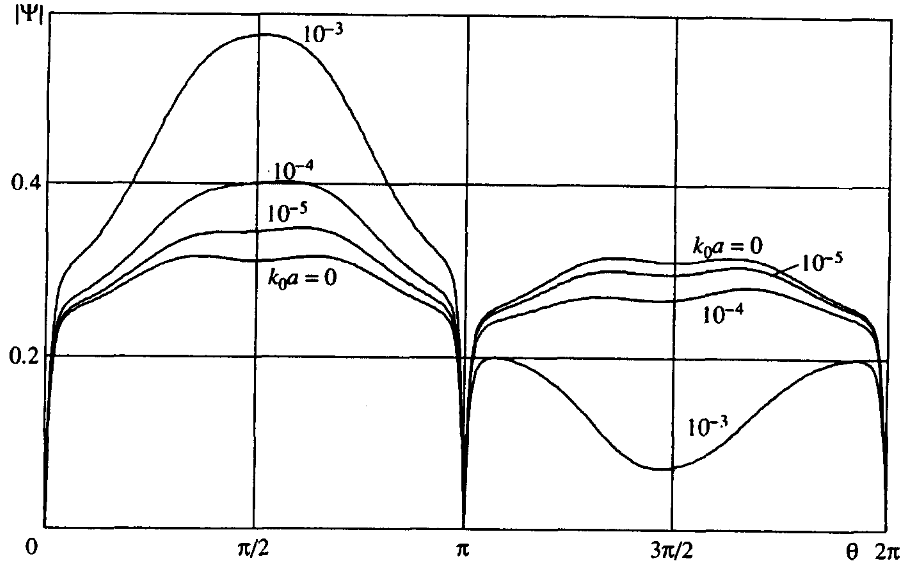

Fig. 1

On the other hand, the effective scattering cross-section can be calculated as the energy scattered at the joint and carried away by the cylindrical wave diverging over the plate and by the four edge waves, i.e.

$$\Sigma = \frac{2\pi}{k_0}\int_0^{2\pi} |\Psi(\vartheta,\vartheta)_0|^2 \, d\vartheta + Q\left(|A_+^e + A_+^o|^2 + |A_+^e - A_+^o|^2 + |A_-^e + A_-^o|^2 + |A_-^e - A_-^o|^2\right) \tag{8.2}$$

where

$$Q = \frac{\varkappa}{k_0 a_+^2(\varkappa)}\left(\frac{a_+^2(\varkappa)}{2b_-(\varkappa)} - \frac{a_-^2(\varkappa)}{2b_+(\varkappa)} + 2(1-\sigma)k_0^2(a_+(\varkappa)b_-(\varkappa) - a_-(\varkappa)b_+(\varkappa))\right)$$

Figure 1 shows the patterns of the cylindrical waves, calculated by means of formula (6.4) for an angle of incidence of 30°. It is assumed that $\sigma = 0.3$. The accuracy of the calculations was monitored by means of the optical theorem (8.1), (8.2). For the patterns presented, the energy balance breaks down by no more than 10^{-6}, which is due to calculation errors. These errors are greatest when $\sigma \approx 0$, since in this case $\varkappa$ is practically identical with k_0.

For a point model of the joint (i.e. when $a = 0$), the scattered field is symmetrical with respect to x and antisymmetrical with respect to y. When $k_0a = 10^{-5}$, the correction to the point model of the joint is considerable. Here the pattern loses symmetry both with respect to x and with respect to y.

The numerical analysis carried out has shown that, if only the leading term and corrections having a logarithmic order of smallness with respect to k_0a are retained in formula (6.4), the optical theorem is satisfied exactly. If, however, terms $O((k_0a)^2)$ are also taken into account, the optical theorem is satisfied approximately.

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